ON THE CHARACTERISTIC EXPONENTS OF THE SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS

(O KHARAKTERISTICHESKIKH POKAZATELIAKH RESHENII LINEINOGO DIFFERENTSIAL'NOGO URAVNENIIA VTOROGO PORIADKA S PERIODICHESKIMI KOEFFITSIENTAMI)

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L.M. MARKHASHOV
(Moscow)

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1. Let the coefficients $q_1(t)$ and $q_2(t)$ of the equation

$$x^{\cdot\cdot} + q_1(t)\,x^{\cdot} + q_2(t)\,x = 0 \tag{1.1}$$

be continuous periodic functions of the real period ω. It is known that

$$\frac{1}{\omega}\int_0^\omega q_1(t)\,dt = -(\lambda_1 + \lambda_2) \tag{1.2}$$

where λ_1 and λ_2 are characteristic exponents of solutions of the equation (1.1). They are real numbers.

The equation (1.2) gives an example of a function F of the coefficients of the equation (1.1), namely, $F \equiv q_1$ which has the following properties: its mean value over the period is a function of the characteristic exponents λ_1 and λ_2 whose structure does not depend on q_1 as a function of time.

The existence of another similar function with a mean value different from $\lambda_1 + \lambda_2$ would permit the evaluation of λ_1 and λ_2 by means of a finite number of standard operations on the coefficients of the equation independently of the particular form the latter might have.

One can show, however, that such a function does not exist.

2. We shall make the statement of the problem more precise. Let us assume that the coefficients $q_1(t)$ and $q_2(t)$ are continuous n-times differentiable functions and such that $\lambda_1 \neq \lambda_2$. Then the independent solutions of the equation (1.1) can be represented in the form (2)

$$x_i = \varphi_i(t)\, e^{\lambda_i t} \tag{2.1}$$

Here, and in what follows, $i = 1, 2$; $\phi_i(t)$ is a periodic function of period ω.

From the existence and uniqueness theorem for the solutions of equation (1.1) and of the equations obtainable form (1.1) through n repeated differentiations, it follows that the functions $\phi_i(t)$ and their derivatives of order up to $n + 2$ are continuous.

We introduce the notation

$$\varphi_i^{(\varkappa)} = \frac{d^{(\varkappa)}\varphi_i}{dt^\varkappa}, \qquad \varphi_i^{(0)} = \varphi_i \qquad (\varkappa = 0, 1, \ldots, n)$$

From the equations, which result from the substitution of each of the solutions (2.1) into equation (1.1), we obtain

$$q_i = a_{i1}\varphi_1^{(2)} + a_{i2}\varphi_2^{(2)} + b_{i0}$$

$$a_{11} = -\frac{\varphi_2}{\Delta}, \qquad a_{12} = \frac{\varphi_1}{\Delta}, \qquad a_{21} = \frac{\varphi_2^{(1)} + \lambda_2\varphi_2}{\Delta}, \qquad a_{22} = -\frac{\varphi_1^{(1)} + \lambda_1\varphi_1}{\Delta}$$

$$\Delta = \varphi_1^{(1)}\varphi_2 - \varphi_1\varphi_2^{(1)} + (\lambda_1 - \lambda_2)\varphi_1\varphi_2, \qquad \frac{\partial b_{i0}}{\partial \varphi_1^{(2)}} = \frac{\partial b_{i0}}{\partial \varphi_2^{(2)}} = 0 \tag{2.2}$$

From these relations it follows that

$$q_i^{(\varkappa)} = a_{i1}\varphi_1^{(\varkappa+2)} + a_{i2}\varphi_2^{(\varkappa+2)} + b_{i\varkappa} \tag{2.3}$$

where the b_{ix} are continuous and depend on $\phi_1^{(x+2)}$, $\phi_2^{(x+2)}$.

We shall consider all possible differentiable functions F_ν of the independent* variables $t, \phi_1, \ldots, \phi_2^{(n+2)}$ periodic relative to the explicitly appearing variable t:

$$F_\nu(t + \omega, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{n+2}) = F_\nu(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)})$$

We shall subject these functions to the following conditions. (1) The mean value of F_ν over the period is a function $c_\nu(\lambda_1, \lambda_2)$ of the characteristic exponents whose form depends on the form of $\phi_i(t)$ as a function

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* It is assumed that the variable t, entering explicitly in F_ν, cannot be expressed in terms of integrals

$$\int_0^t G\,[\varphi_i^{\varkappa}(t)]\,dt$$

where G is an arbitrary integrable, nonconstant function of the variables $\phi_i^{(x)}$.

of t:

$$\frac{1}{\omega}\int_0^{\omega} F_\nu[t, \lambda_1, \lambda_2, \varphi_1(t), \ldots, \quad \varphi_2^{(n+2)}(t)]\, dt = c_\nu(\lambda_1, \lambda_2) \tag{2.4}$$

(2) The arguments of the function F_ν can be grouped so that in view of (2.3) for arbitrary $\phi_i^{(x)}$ the following identity holds

$$F_\nu(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)}) \equiv F_\nu'(t, q_1, \ldots, q_2^{(n)}) \tag{2.5}$$

The F_ν' do not depend explicitly on λ_i and $\phi_i^{(x)}$.

The functions that satisfy conditions (2.4) and (2.5) we include in the class $\{F\}$. They are periodic in t and are such that the integral

$$\frac{1}{\omega}\int_0^{\omega} F[t, q_1(t), \ldots, q_2^{(n)}(t)]\, dt$$

is a function of the characteristic exponents whose structure depends only on the choice of F. The arguments t, $q_i^{(x)}$ of F are independent. The following theorem holds.

Theorem. If the n-times differentiable periodic functions are not all identical constants, and are such that $\lambda_1 \neq \lambda_2$, then for an arbitrary function F of the class $\{F\}$ the following identity is valid

$$\frac{1}{\omega}\int_0^{\omega} F[t, q_1(t), \ldots, q_2^{(n)}(t)]\, dt = \mu_1(\lambda_1 + \lambda_2) + \mu$$

where μ_1 and μ are constants independent of λ_i.

It follows from the theorem that the characteristic exponents of the solution of the equation (1.1) cannot be expressed in terms of a finite relationship between the mean values of the functions of the class $\{F\}$.

Below we derive two lemmas with the aid of which the conditions (2.4) and (2.5) (that restrict the class of functions F_ν) are formulated in a manner convenient for the proof of the theorem.

Lemma 1. In order that the function F_ν satisfy the condition (2.4) it is necessary and sufficient that the following equation be satisfied identically in the $\phi_i^{(x)}$:

$$\Psi_{n+2}^{(i)}(F_\nu) \equiv \frac{\partial F_\nu}{\partial \varphi_i} - \frac{d}{dt}\left(\frac{\partial F_\nu}{\partial \varphi_i^{(1)}}\right) + \ldots + (-1)^{n+2}\frac{d^{(n+2)}}{dt^{n+2}}\left(\frac{\partial F_\nu}{\partial \varphi_i^{(n+2)}}\right) = 0 \tag{2.6}$$

Proof. Without loss of generality we assume that $\omega = 2\pi$. The functions $\phi_i^{(x)}$ admit expansions into convergent Fourier series:

$$\varphi_i = a_0^{(i)} + \sum_{k=1}^{\infty} (a_k^{(i)} \cos kt + b_k^{(i)} \sin kt)$$

$$\varphi_i^{(1)} = \sum_{k=1}^{\infty} k\,(-a_k^{(i)} \sin kt + b_k^{(i)} \cos kt)$$

$$\varphi_i^{(2)} = -\sum_{k=1}^{\infty} k^2\,(a_k^{(i)} \cos kt + b_k^{(i)} \sin kt)$$

The functional

$$J = \frac{1}{2\pi}\int_0^{2\pi} F_\nu(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)})\,dt \quad \textbf{etc.}$$

will not depend on the form of the function $\phi_i^{(x)}$ if, and only if,

$$\frac{\partial J}{\partial a_k^{(i)}} = \frac{\partial J}{\partial b_k^{(i)}} = \frac{\partial J}{\partial a_0^{(i)}} = 0$$

We have

$$\frac{\partial J}{\partial a_k^{(i)}} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial F_\nu}{\partial \varphi_i}\cos kt + \frac{\partial F_\nu}{\partial \varphi_i^{(1)}}(-k\sin kt) + \frac{\partial F_\nu}{\partial \varphi_i^{(2)}}(-k^2\cos kt) + \ldots\right]dt$$

$$\frac{\partial J}{\partial b_i^{(1)}} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial F_\nu}{\partial \varphi_i}\sin kt + \frac{\partial F_\nu}{\partial \varphi_i^{(1)}}k\cos kt + \frac{\partial F_\nu}{\partial \varphi_i^{(2)}}(-k^2\sin kt) + \ldots\right]dt$$

Integrating by parts the appropriate number of times, and taking into account the periodicity of the function F_ν, we obtain

$$\frac{\partial J}{\partial a_k^{(i)}} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_{n+2}^{(i)}(F_\nu)\cos kt\,dt = 0, \qquad \frac{\partial J}{\partial b_k^{(i)}} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_{n+2}^{(i)}(F_\nu)\sin kt\,dt = 0$$

Furthermore,

$$\frac{\partial J}{\partial a_0^{(i)}} = \frac{1}{2\pi}\int_0^{2\pi} \Psi_{n+2}^{(i)}(F_\nu)\,dt = 0$$

This establishes the lemma.

We note that the content of the proved lemma does not coincide with the content of a very similar variational problem. Therefore the validity of this lemma is not a consequence of any fact known in the calculus of variation.

Consequence. It is not difficult to establish that equation (2.6) is satisfied by any function F_ν representable in the form

$$F_\nu(t, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \frac{d}{dt} f(t, \varphi_1, \ldots, \varphi_2^{(n+1)})$$

where f is an arbitrary differentiable function periodic in t.

Lemma 2. Let r_m be the rank of the functional determinant

$$M = \begin{Vmatrix} \frac{\partial F_\nu}{\partial \varphi_1} & \frac{\partial F_\nu}{\partial \varphi_2} & \cdots & \frac{\partial F_\nu}{\partial \varphi_2^{(n+2)}} \\ \frac{\partial q_1}{\partial \varphi_1} & \frac{\partial q_1}{\partial \varphi_2} & \cdots & \frac{\partial q_1}{\partial \varphi_2^{(n+2)}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial q_2^{(n)}}{\partial \varphi_1} & \frac{\partial q_2^{(n)}}{\partial \varphi_2} & \cdots & \frac{\partial q_2^{(n)}}{\partial \varphi_2^{(n+2)}} \end{Vmatrix}$$

The function F_ν will satisfy condition (2.5) if and only if $r_m \leqslant 2(n+1)$ for arbitrary $\phi_i^{(x)}$.

The validity of the lemma follows from a known theorem in analysis [3].

We note that this lemma does not have a local character by the very nature of the condition (2.5) which must hold everywhere.

3. We now proceed with the proof of the theorem.

We shall first show that the function F_ν which is subject to condition (2.4) must be representable in the form

$$F_\nu = u_n\varphi_1^{(n+2)} + v_n\varphi_2^{(n+2)} + A_n \tag{3.1}$$

with

$$\frac{\partial u_n}{\partial \varphi_2^{(n+1)}} = \frac{\partial v_n}{\partial \varphi_1^{(n+1)}}, \qquad \frac{\partial u_n}{\partial \varphi_i^{(n+2)}} = \frac{\partial v_n}{\partial \varphi_i^{(n+2)}} = \frac{\partial A_n}{\partial \varphi_i^{(n+2)}} = 0 \tag{3.2}$$

Indeed, since the function F_ν contains by hypothesis only derivatives of order not higher than $n+2$, the coefficients of the higher order derivatives in the operator (2.6) must vanish.

For the coefficients of $\phi_i^{2(n+2)}$ we have

$$\partial^2 F / \partial\varphi_i^{(n+2)}\partial\varphi_j^{(n+2)} = 0 \qquad (j = 1,\ 2) \tag{3.3}$$

Whence,

$$F_\nu = u_n\varphi_1^{(n+2)} + v_n\varphi_2^{(n+2)} + A_n$$

wherein the second condition (3.2) is satisfied.

The terms containing derivatives of order $2(n+2) - 1$ are generated by the last two terms of the operator (2.4):

$$(-1)^{n+1}\frac{d^{(n+1)}}{dt^{n+1}}\left(\frac{\partial F_\nu}{\partial \varphi_i^{(n+1)}}\right) \equiv$$

$$\equiv (-1)^{n+1}\frac{d^{(n+1)}}{dt^{n+1}}\left(\frac{\partial u_n}{\partial \varphi_i^{(n+1)}}\,\varphi_1^{(n+2)} + \frac{\partial v_n}{\partial \varphi_i^{(n+1)}}\,\varphi_2^{(n+2)} + \frac{\partial A_n}{\partial \varphi_i^{(n+1)}}\right)$$

$$(-1)^{n+2}\frac{d^{(n+1)}}{dt^{n+1}}\left(\frac{\partial z_i}{\partial \varphi_1^{(n+1)}}\varphi_1^{(n+2)}+\frac{\partial z_i}{\partial \varphi_2^{(n+1)}}\varphi_2^{(n+1)}+\ldots\right)$$

Here,

$$z_i=\begin{cases} u_n & (i=1) \\ v_n & (i=2)\end{cases}$$

The coefficient of the first power of the function $\phi_1^{(2n+3)}$ is

$$\frac{\partial u_n}{\partial \varphi_i^{(n+1)}}-\frac{\partial z_i}{\partial \varphi_1^{(n+1)}}=0$$

Setting $i = 2$ in this expression we obtain equation (3.2).

For the sake of explicitness let us restrict ourselves to the case when $n = 1$. It is the simplest case but contains all the characteristics of the general one.

We note first of all that the determinant

$$D=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}=\Delta^{-1}=e^{-(\lambda_1+\lambda_2)t}\begin{vmatrix} x_1^{\cdot} & x_1 \\ x_2^{\cdot} & x_2\end{vmatrix}^{-1}$$

is bounded and different from zero because of the linear independence and continuity of the solutions of equation (1.1).

We require that the functions F_ν also satisfy the condition (2.5). According to lemma 2, with $n = 1$, we must have $r_m \leqslant 4$. Hence one can find at least one set of values $\mu_0, \mu_1, \ldots, \mu_4$, not all zero, such that

$$\mu_0\frac{\partial F_\nu}{\partial \varphi_i^{(j)}}+\mu_1\frac{\partial q_1}{\partial \varphi_i^{(j)}}+\mu_2\frac{\partial q_2}{\partial \varphi_i^{(j)}}+\mu_3\frac{\partial q_1^{(1)}}{\partial \varphi_i^{(j)}}+\mu_4\frac{\partial q_2^{(1)}}{\partial \varphi_i^{(j)}}=0 \quad (j=0,1,2,3) \qquad (3.4)$$

These equations must be satisfied identically in $\phi_i^{(j)}$. We note that $\mu_0 \neq 0$, for otherwise there would exist a linear relation between the rows of the matrix M. But this is impossible since the fourth-order determinant of the matrix M, standing in the lower right-hand corner, is different from zero:

$$\frac{\partial(q_1, q_2, q_1^{(1)}, q_2^{(1)})}{\partial(\varphi_1^{(2)}, \varphi_2^{(2)}, \varphi_1^{(3)}, \varphi_2^{(3)})}=\begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \cdot & \cdot & a_{11} & a_{12} \\ \cdot & \cdot & a_{21} & a_{22}\end{vmatrix}=D^2\neq 0$$

Thus, without loss of generality, $\mu_0 = 1$.

Taking into account the linearity of the functions $F_\nu, q_1, \ldots, q_2^{(1)}$ relative to the $\phi_i^{(3)}$, and making use of the condition $D \neq 0$, we can easily prove by differentiation equation (3.4) with respect to $\phi_i^{(3)}$ that

$$\frac{\partial \mu_3}{\partial \varphi_i^{(3)}} = \frac{\partial \mu_1}{\partial \varphi_i^{(3)}} = 0.$$

$$\mu_1 = \mu_{11}\varphi_1^{(3)} + \mu_{12}\varphi_2^{(3)} + \mu_1', \qquad \mu_2 = \mu_{21}\varphi_1^{(3)} + \mu_{22}\varphi_2^{(3)} + \mu_2'$$

where $\mu_{i\rho}$ and μ_1' are quantities independent of $\phi_i^{(3)}$.

Substituting the values μ_1 and μ_2 into (3.4), equating to zero the coefficients of $\phi_i^{(3)}$ and taking into account the equations (2.3) and (3.1), we obtain

$$\begin{gathered}
\frac{\partial u_1}{\partial \varphi_1} + \mu_3 \frac{\partial a_{11}}{\partial \varphi_1} + \mu_4 \frac{\partial a_{21}}{\partial \varphi_1} + \mu_{11} \frac{\partial q_1}{\partial \varphi_1} + \mu_{21} \frac{\partial q_2}{\partial \varphi_1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial u_1}{\partial \varphi_2^{(2)}} + \mu_3 \frac{\partial a_{11}}{\partial \varphi_2^{(2)}} + \mu_4 \frac{\partial a_{21}}{\partial \varphi_2^{(2)}} + \mu_{11} \frac{\partial q_1}{\partial \varphi_2^{(2)}} + \mu_{21} \frac{\partial q_2}{\partial \varphi_2^{(2)}} = 0
\end{gathered} \tag{3.5}$$

$$\begin{gathered}
\frac{\partial v_1}{\partial \varphi_1} + \mu_3 \frac{\partial a_{12}}{\partial \varphi_1} + \mu_4 \frac{\partial a_{22}}{\partial \varphi_1} + \mu_{12} \frac{\partial q_1}{\partial \varphi_1} + \mu_{22} \frac{\partial q_2}{\partial \varphi_1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial v_1}{\partial \varphi_2^{(2)}} + \mu_3 \frac{\partial a_{12}}{\partial \varphi_2^{(2)}} + \mu_4 \frac{\partial a_{22}}{\partial \varphi_2^{(2)}} + \mu_{12} \frac{\partial q_1}{\partial \varphi_2^{(2)}} + \mu_{22} \frac{\partial q_2}{\partial \varphi_2^{(2)}} = 0
\end{gathered} \tag{3.6}$$

$$u_1 + \mu_3 a_{11} + \mu_4 a_{21} = 0, \qquad v_1 + \mu_3 a_{12} + \mu_4 a_{22} = 0 \tag{3.7}$$

The system of equations (3.7) is obtained directly as a statement of the fact that the elements of the last two columns of the matrix M are linearly independent.

The group of equations, obtainable from (3.4) by equating to zero the sum of the terms independent of $\phi_i^{(3)}$, has not been written out. Eliminating u_1 from the system (3.5), and v_1 from the system (3.6), we obtain with the aid of (3.7) the next set of equations

$$\begin{gathered}
-a_{11} \frac{\partial \mu_3}{\partial \varphi_1} - a_{21} \frac{\partial \mu_4}{\partial \varphi_1} + \mu_{11} \frac{\partial q_1}{\partial \varphi_1} + \mu_{21} \frac{\partial q_2}{\partial \varphi_1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{11} \frac{\partial \mu_3}{\partial \varphi_2^{(2)}} - a_{21} \frac{\partial \mu_4}{\partial \varphi_2^{(2)}} + \mu_{11} \frac{\partial q_1}{\partial \varphi_2^{(2)}} + \mu_{21} \frac{\partial q_2}{\partial \varphi_2^{(2)}} = 0 \\
-a_{12} \frac{\partial \mu_3}{\partial \varphi_1} - a_{22} \frac{\partial \mu_4}{\partial \varphi_1} + \mu_{12} \frac{\partial q_1}{\partial \varphi_1} + \mu_{22} \frac{\partial q_2}{\partial \varphi_1} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{12} \frac{\partial \mu_3}{\partial \varphi_2^{(2)}} - a_{22} \frac{\partial \mu_4}{\partial \varphi_2^{(2)}} + \mu_{12} \frac{\partial q_1}{\partial \varphi_2^{(2)}} + \mu_{22} \frac{\partial q_2}{\partial \varphi_2^{(2)}} = 0
\end{gathered}$$

From this follows the existence of two identities in $\phi_1, \ldots, \phi_2^{(2)}$: $X_1(\mu_3, \mu_4, q_1, a_2, t) = 0$, $X_2(\mu_3, \mu_4, q_1, q_2, t) = 0$.

Furthermore, for arbitrary finite q_i and t,

$$\frac{\partial (X_1, X_2)}{\partial (\mu_3, \mu_4)} \sim D \neq 0$$

The equations $X_i = 0$ are therefore solvable for μ_0 and μ_4:

$$\mu_3 = \mu_3 (q_1, q_2, t), \qquad \mu_4 = \mu_4 (q_1, q_2, t)$$

The substitution of u_1 and v_1 from equations (3.7) into the equation (3.2) after some simple reductions with the aid of (2.2), yields the following equations

$$D \left(\frac{\partial \mu_3}{\partial q_2} - \frac{\partial \mu_4}{\partial q_1} \right) = 0, \quad \text{or} \quad \frac{\partial \mu_3}{\partial q_2} = \frac{\partial \mu_4}{\partial q_1}$$

Hence, there exists a differentiable function $\Phi_1(q_1, q_2, t)$ such that $\mu_3 = \partial \Phi_1 / \partial q_1$, $\mu_4 = \partial \Phi_1 / \partial q_2$. On the basis of (3.2) we have

$$u_1 = \partial f_1 / \partial \varphi_1^{(2)}, \qquad v_1 = \partial f_1 / \partial \varphi_2^{(2)}$$

It follows, therefore, from equation (3.7) that

$$\frac{\partial}{\partial \varphi_i^{(2)}} [f_1 - \Phi_1 (q_1, q_2, t)] = 0$$

where f_1 is a differentiable function of the variables $t, \phi_1, \ldots, \phi_2^{(2)}, \lambda_1, \lambda_2$.

Making use of the last relations, we can reduce the function $F = u_1\phi_1^{(3)} + v_1\phi_2^{(3)} + A_1$ to the form

$$F_\nu = \frac{d}{dt} \Phi_1 (q_1, q_2, t) + B_1 (t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(2)})$$

where B_1 is some differentiable function not containing $\phi_1^{(3)}$.

In the general case ($n > 1$), a function F_ν, which is subject to the conditions (2.4) and (2.5), must necessarily be representable in the form

$$F_\nu (t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \frac{d}{dt} \Phi_n (t, q_1, \ldots, q_2^{(n-1)}) + \\ + B_n (t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+1)})$$

In accordance with lemma 1, $\Psi_{n+2}^{(i)}(F_\nu) = 0$. Because of the linearity of the operator Ψ_{n+2} (the superscript i has been dropped), we have the following relation:

$$\Psi_{n+2} (F_\nu) = \Psi_{n+2} (d\Phi_n / dt) + \Psi_{n+2} (B_n)$$

In consequence of Lemma 1, we also have

$$\Psi_{n+2} (d\Phi_n / dt) = 0$$

Therefore, $\Psi_{n+2}(B_n) = 0$. But $\partial B_n/\partial\phi^{(n+2)} = 0$, and the order of the operator has been lowered by one: $\Psi_{n+1}(B_n) = 0$. In entirely analogous manner one obtains

$$B_n(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+1)}) = \frac{d}{dt}\Phi_{n-1}(t, q_1, \ldots, q_2^{(n-2)}) + \\ + B_{n-1}(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n)}), \qquad \Psi_n(B_{n-1}) = 0$$

Thus,

$$F_\nu(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \frac{d}{dt}\sum_{k=1}^{n}\Phi_k + B_1(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(2)}) =$$

$$= \frac{d}{dt}\Phi(t, q_1, \ldots, q_2^{(n-1)}) + B_1(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_3^{(2)}) \qquad \Psi_2(B_1) = 0$$

Here $\Phi = \Phi_1 + \ldots + \Phi_n$.

The function B_1 must also satisfy the condition (2.5). In accordance with lemma 2, the corresponding equations ($n = 0$), in which

$$B_1 = u_0(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(1)})\varphi_1^{(2)} + v_0(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(1)})\varphi_2^{(2)} + \\ + A_0(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(1)})$$

are obtained in the form:

$$\frac{\partial u_0}{\partial\varphi_1} + \mu_1\frac{\partial a_{11}}{\partial\varphi_1} + \mu_2\frac{\partial a_{21}}{\partial\varphi_1} = 0, \ldots, \qquad \frac{\partial u_0}{\partial\varphi_2^{(1)}} + \mu_1\frac{\partial a_{11}}{\partial\varphi_2^{(1)}} + \mu_2\frac{\partial a_{21}}{\partial\varphi_2^{(1)}} = 0 \quad (3.8)$$

$$\frac{\partial v_0}{\partial\varphi_1} + \mu_1\frac{\partial a_{12}}{\partial\varphi_1} + \mu_2\frac{\partial a_{22}}{\partial\varphi_1} = 0, \ldots, \qquad \frac{\partial v_0}{\partial\varphi_2^{(1)}} + \mu_1\frac{\partial a_{12}}{\partial\varphi_2^{(1)}} + \mu_2\frac{\partial a_{22}}{\partial\varphi_2^{(1)}} = 0 \quad (3.9)$$

$$u_0 + \mu_1 a_{11} + \mu_2 a_{21} = 0, \qquad v_0 + \mu_1 a_{12} + \mu_2 a_{22} = 0 \quad (3.10)$$

$$\frac{\partial A_0}{\partial\varphi_1} + \mu_1\frac{\partial b_{10}}{\partial\varphi_1} + \mu_2\frac{\partial b_{20}}{\partial\varphi_1} = 0, \ldots, \qquad \frac{\partial A_0}{\partial\varphi_2^{(1)}} + \mu_1\frac{\partial b_{10}}{\partial\varphi_2^{(1)}} + \mu_2\frac{\partial b_{20}}{\partial\varphi_2^{(1)}} = 0 \quad (3.11)$$

The last set of equations is obtained by equating to zero all the terms free of $\phi_i^{(2)}$ in the equations (3.4) written out for the case $n = 0$. Eliminating u_0 from (3.8) and v_0 from (3.9) with the aid of equations (3.1), we obtain

$$a_{11}\frac{\partial\mu_1}{\partial\varphi_1} + a_{21}\frac{\partial\mu_2}{\partial\varphi_1} = 0, \ldots, \qquad a_{11}\frac{\partial\mu_1}{\partial\varphi_2^{(1)}} + a_{21}\frac{\partial\mu_2}{\partial\varphi_2^{(1)}} = 0$$

$$a_{12}\frac{\partial\mu_1}{\partial\varphi_1} + a_{22}\frac{\partial\mu_2}{\partial\varphi_1} = 0, \ldots, \qquad a_{12}\frac{\partial\mu_1}{\partial\varphi_2^{(1)}} + a_{22}\frac{\partial\mu_2}{\partial\varphi_2^{(1)}} = 0$$

Pairing off equations of the last two systems and taking into account the fact that $D \neq 0$, we obtain

$$\frac{\partial\mu_1}{\partial\varphi_i^{(j)}} = \frac{\partial\mu_2}{\partial\varphi_i^{(j)}} = 0 \qquad (j = 1, 2)$$

and, hence, $\mu_1 = \mu_1(t)$, $\mu_2 = \mu_2(t)$.

The system (3.11) reduces to a single equation

$$A_0 + \mu_1 b_{10} + \mu_2 b_{20} = \psi(t)$$

From the condition $\partial u_0 / \partial \phi_2^{(1)} = \partial v_0 / \partial \phi_1^{(1)}$, in view of relations (3.10), (2.2) and since $D \neq 0$, we find that $\mu_2 = 0$. The systems (3.8), (3.9) and (3.11) reduce finally to three equations:

$$A_0 + \mu_1 b_{10} = \psi(t), \qquad u_0 + \mu_1 a_{11} = 0, \qquad v_0 + \mu_1 a_{12} = 0$$

Thus,

$$B_1 = u_0 \varphi_1^{(2)} + u_0 \varphi_2^{(2)} + A_0 = \psi(t) - \mu_1 (a_{11}\varphi_1^{(2)} + a_{12}\varphi_2^{(2)} + b_{10}) = \psi(t) - \mu_1 q_1$$

We next show that μ_1 = const. From the equation $\Psi_2(B_1) = 0$ it follows that

$$B_1 = \frac{d}{dt} \zeta(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(1)}) + \zeta_1(t)$$

where ζ and ζ_1 are some differentiable functions independent of each other. But

$$\mu_1 q_1 = \mu_1 \frac{d}{dt} [(\lambda_1 + \lambda_2) t + \ln \Delta]$$

On the other hand,

$$\mu_1 q_1 = -\frac{d}{dt} \left(\zeta + \int \zeta_1 dt - \int \psi dt \right)$$

Therefore, the quantity

$$\mu_1 \frac{d}{dt} [(\lambda_1 + \lambda_2) t + \ln \Delta]$$

must be a total derivative with respect to t. This, however, is possible only if μ_1 = const. Thus,

$$F_v(t, \lambda_1, \lambda_2, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \\ = \frac{d}{dt} \Phi(t, q_1, \ldots, q_2^{(n-1)}) - \mu_1 q_1 + \psi(t) \qquad (\mu_1 = \text{const})$$

is a function of the class $\{F\}$ of the most general type, and

$$\frac{1}{2\pi} \int_0^{2\pi} F dt = \frac{1}{2\pi} \Phi \Big|_0^{2\pi} - \frac{\mu_1}{2\pi} \int_0^{2\pi} q_1 dt + \frac{1}{2\pi} \int_0^{2\pi} \psi(t) dt = \mu_1 (\lambda_1 + \lambda_2) + \mu$$

This proves the theorem.

4. By dropping in the expression for the function F the unessential term $\psi(t)$, one may write the following equation, on the basis of what has just been proved.

$$\frac{d}{dt} f(t, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \frac{d}{dt} \Phi(t, q_1, \ldots, q_2^{(n-1)}) - \mu_1 q_1$$

Integrating this expression, setting $f = \ln f'$, $\Phi = \ln \Phi'$, and exponentiating the result, we obtain

$$f'(t, \varphi_1, \ldots, \varphi_2^{(n+2)}) = \Phi'(t, q_1, \ldots, q_2^{(n-1)}) \exp\left(-\mu_1 \int q_1 dt\right)$$

Such a representation of functions in terms of the fundamental solutions of the equation (1.1) is found in connection with a theorem due to Appel [1].

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